

# Decomposition of stochastic flows generated by Stratonovich SDEs with jumps

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## Abstract

Consider a manifold  $M$  endowed locally with a pair of complementary distributions  $\Delta^H \oplus \Delta^V = TM$  and let  $\text{Diff}(\Delta^H, M)$  and  $\text{Diff}(\Delta^V, M)$  be the corresponding Lie subgroups generated by vector fields in the corresponding distributions. We decompose a stochastic flow with jumps, up to a stopping time, as  $\varphi_t = \xi_t \circ \psi_t$ , where  $\xi_t \in \text{Diff}(\Delta^H, M)$  and  $\psi_t \in \text{Diff}(\Delta^V, M)$ . Our main result provides Stratonovich stochastic differential equations with jumps for each of these two components in the corresponding infinite dimensional Lie groups. We present an extension of the Itô-Ventzel-Kunita formula for stochastic flows with jumps generated by classical Marcus equation (as in Kurtz, Pardoux and Protter [11]). The results here correspond to an extension of Catuogno, da Silva and Ruffino [4], where this decomposition was studied for the continuous case.

**Key words:** semimartingales with jumps, stochastic flows, smooth distributions, decomposition of flows, group of diffeomorphisms, Marcus equation.

**MSC2010 subject classification:** 60H10, 58J65, 58D05.

## 1 Introduction

Let  $M$  be a compact differentiable manifold, with  $\text{Diff}(M)$  the corresponding infinite dimensional Lie group of diffeomorphisms of  $M$  generated by the Lie algebra of smooth vector fields. Consider a stochastic flow  $\varphi_t$  of local diffeomorphisms in  $M$ , generated by a Stratonovich SDE driven by a semimartingale  $Z_t$  with jumps. The question we address in this paper is the possibility of decomposing  $\varphi_t$  in components that belong to specific subgroups of  $\text{Diff}(M)$ , which provide dynamical or geometrical information of the system. In the continuous case, decompositions of this kind for stochastic flows has been studied in the literature, with many distinct frameworks and with different aimed subgroups. Among others see Bismut [2], Kunita [8], [9], Ming Liao [12] and some of our previous work [3], [6], [17]. In the last few papers mentioned, for example, geometrical conditions on a Riemannian manifold have been stated to guarantee the existence of the decomposition where the first component lies in the subgroup of isometries or affine transformations.

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Our geometrical framework here is the following: consider a manifold  $M$  equipped with a pair of complementary distributions  $\Delta^H$  and  $\Delta^V$  (horizontal and vertical, respectively) in the sense of differentiable sections in Grassmanian bundles of  $M$ . For each smooth vector field  $X$  in  $M$ , denote by  $\exp\{tX\} \in \text{Diff}(M)$  the associated flow. Given a distribution  $\Delta$  in  $M$ , we shall denote by  $\text{Diff}(\Delta, M)$  the closure of the subgroup of diffeomorphisms generated by exponentials of vector fields in  $\Delta$ , precisely:

$$\text{Diff}(\Delta, M) = \text{cl}\left\{ \exp\{t_1 X_1\} \circ \dots \circ \exp\{t_n X_n\}, \text{ with } X_i \in \Delta, t_i \in \mathbf{R}, \text{ for all } n \in \mathbf{N} \right\}.$$

The differentiable structure of infinite dimensional Lie group of diffeomorphisms of a compact manifold  $M$  is classically given by maps in the locally convex space of smooth vector fields in  $M$ , more precisely, maps in an ILB-space, see Omori [15], Milnor [13] and more recently Neeb [14]. So that the closure above is taken with respect to this topology. Existence and uniqueness of solution of an ODE in an ILB-spaces with smooth vector fields has being guaranteed by [15, Thm 7.2, Chap.1]. Hence, solutions of Stratonovich differential equations in these spaces can be obtained via support theorem if the noise is finite dimensional, see e.g. Twardowska [19]. Note that, considering the limits, by Baker-Campbell-Hausdorff formula, this subgroup includes flows of diffeomorphisms associated to  $\text{Lie}(\Delta)$ , the Lie algebra generated by  $\Delta$ . Precisely, we have that

$$\text{Diff}(\Delta, M) = \text{Diff}(\text{Lie}(\Delta), M).$$

Using this notation, the stochastic flow  $\varphi_t$  belongs to  $\text{Diff}(TM, M)$ , which is a connected subgroup of  $\text{Diff}(M)$ . Again,  $\text{Diff}(TM, M)$  contains the Lie subgroups  $\text{Diff}(\Delta^H, M)$  and  $\text{Diff}(\Delta^V, M)$ . If both distributions  $\Delta^H$  and  $\Delta^V$  are involutive, locally the intersection of these two subgroups reduces to the identity, and the elements of each of these subgroups send leaves of the corresponding foliation into themselves. This also implies that if the decomposition exists, it is unique.

In this article we want to decompose a stochastic flow, up to a stopping time, as  $\varphi_t = \xi_t \circ \psi_t$ , where  $\xi_t \in \text{Diff}(\Delta^H, M)$  and  $\psi_t \in \text{Diff}(\Delta^V, M)$ . Our main result provides Stratonovich stochastic differential equations with jumps for each of these two components in the corresponding infinite dimensional Lie groups (Theorem 4.5). One of the key points on obtaining the equations for each component  $\xi_t$  and  $\psi_t$  is an extension of the classical Itô-Ventzel-Kunita formula for discontinuous càdlàg processes, which are also generated by SDEs with jumps (Theorem 3.2); here, we generalize the definition of the integration with respect to a change of variables in Kurtz, Pardoux and Protter [11].

A motivation for this decomposition appears in many dynamical systems. For instance, in a foliated space (say, assuming that the horizontal distribution is integrable) the decomposition measures how far the trajectories are from being foliation-preserving: namely, the closer the second component  $\psi_t$  is to the identity, closer the system is to be (horizontal) foliation-preserving. Vice-versa, changing the order of the vertical and horizontal distributions and the order of the components in the decomposition, the same interpretation holds if the vertical component is integrable. For example: a principal bundle with a connection is a natural state space with this feature, where the vertical distribution is determined by the fibres and the horizontal distribution is given by the geometry. The dynamics in

averaging principles for transversal perturbation in foliated systems also illustrates the relevance of this decomposition, see e.g. the Liouville torus in Hamiltonian systems in Li [10], or in a general compact foliation in Gargate and Ruffino [7].

In this paper, we consider the Stratonovich SDEs driven by semimartingales with jumps  $Z_t$ , in the sense of the so called Marcus equation (see e.g. [11]). This means, among other aspects (recalled in Section 2), that the jumps of the solution occur in the direction of the deterministic flow generated by the corresponding vector fields. This property guarantees that if the manifold  $M$  is embedded in an Euclidean space and the vector fields are in  $TM$  then, after the jumps, the solution flow remains in  $M$ . Also, we assume that the number of jumps in  $Z_t$  is finite in a bounded interval of time a.s.. Hence, it includes as a possible integrator the so called Lévy–jump diffusion (see Applebaum [1]), but not Lévy process in general. This idea of finite jumps in bounded intervals has parallel in the theory of chain control sets, see Colonius and Kliemann [5], Patrão and San Martin [18], and references therein.

The paper is organized as follows: In Section 2, we describe some basic properties and definitions of Stratonovich SDE driven by a semimartingale with jumps, in the sense of Marcus equation. In Section 3, our extension of the Itô–Ventzel–Kunita formula is presented, for two stochastic flows generated by Marcus equations with respect to the same semimartingale with jumps. Finally, in section 4, we deal with the decomposition of the stochastic flow  $\varphi_t = \xi_t \circ \psi_t$ , obtaining the corresponding SDEs for each component. The results here correspond to an extension of the article Catuogno, da Silva and Ruffino [4], where the same decomposition is studied for continuous flows.

## 2 Stratonovich SDEs with jumps

For reader's convenience, we recall the definition of a Stratonovich SDE driven by a semimartingale with jumps in  $\mathbf{R}^d$ , in the sense of Marcus equation (for more details, see e.g. Kurtz, Pardoux e Protter [11]):

$$dx_t = \sum_{i=1}^m X^i(x_t) \diamond dZ_t^i, \quad (1)$$

with initial condition  $x(0) = x_0$ . The solution, in its integral form is:

$$x_t = x_0 + \sum_{i=1}^m \int_0^t X^i(X_s) \diamond dZ_s^i =: x_0 + \int_0^t X(x_s) \diamond dZ_s, \quad (2)$$

where  $x_t$  is an adapted stochastic process taking values in  $\mathbf{R}^d$ , the integrator  $\{Z_s^i : s \geq 0\}$  is a semimartingale with jumps and  $X^i$  are vector fields in  $\mathbf{R}^d$  for all  $i \in \{1, \dots, m\}$ . For sake of simplicity we use the matrix notation  $X = (X^1, \dots, X^m)$  and  $Z = (Z^1, \dots, Z^m)$ . The solution is interpreted as a stochastic process that satisfies the equation:

$$\begin{aligned} x_t = x_0 &+ \int_0^t X(x_{s-}) dZ_s + \frac{1}{2} \int_0^t X' X(x_s) d[Z, Z]_s^c \\ &+ \sum_{0 < s \leq t} \{ \phi(X \Delta Z_s, x_{s-}) - x_{s-} - X(x_{s-}) \Delta Z_s \}, \end{aligned} \quad (3)$$

in the following sense: the first term on the right hand side of (3) is a standard Itô integral of the predictable process  $X(x_{t-})$  with respect to the semimartingale  $Z_t$ . The second term is a Stieltjes integral with respect to the continuous part of the quadratic variation of  $Z_t$ . In the third term:  $\phi(X\Delta Z_s, x_{s-})$  indicates the solution at time one of the ODE generated by the vector field  $X\Delta Z_s$  and initial condition  $x_{s-}$ . Thus, the jumps of this equation occurs in deterministic directions. Regularity conditions on vector fields  $X$  implies that there exists a unique stochastic flow of diffeomorphisms  $\varphi_t$ . Conditions on the derivatives of the vector fields guarantee that there exists a stochastic flow of local diffeomorphisms. Moreover, for an embedded submanifold  $M$  in an Euclidean space, if the vector fields of the equation (1) are in  $TM$  then, for each initial condition on  $M$ , the solution stays in  $M$  a.s., see [11].

Let  $Y$  be smooth vector fields in  $\mathbf{R}^d$ , and consider  $G_t$  a  $d$ -dimensional process given by  $dG_t = Y(G_t) \diamond dZ_t$ . If  $H \in C^1(\mathbf{R}^d; \mathbf{R}^m)$  we recall the definition of the Stratonovich integral of  $H(G_t)$  with respect to  $Z_t$  (Definition 4.1 in [11]):

$$\begin{aligned} \int_0^t H(G_s) \diamond dZ_s &:= \int_0^t H(G_{s-}) dZ_s + \frac{1}{2} \text{Tr} \int_0^t H'(G_s) d[Z, Z]_s^c Y(G_s)^t \\ &+ \sum_{0 \leq s \leq t} \left( \int_0^1 [H(\phi(Y\Delta Z_s, G_{s-}, u)) - H(G_{s-})] du \right) \Delta Z_s, \end{aligned} \quad (4)$$

where  $\phi(Y\Delta Z_s, G_{s-}, u)$  has the same interpretation as before: it is the solution of the ODE generated by the vector field  $Y\Delta Z_s$ , and initial condition  $G_{s-}$ , at time  $t = u$ . More specifically, given  $t_0 > 0$ , the jump of the integral on the left hand side of equation (4) is given by:

$$\Delta \left( \int_0^t H(G_s) \diamond dZ_s \right)_{t_0} = \left( \int_0^1 H(\phi(Y\Delta Z_{t_0}, G_{t_0-}, u)) du \right) \Delta Z_{t_0}.$$

The dynamical interpretation of the expression above is that, after opening a unit interval with a ‘fictitious curve’ which connects  $G_{t_0-}$  and  $G_{t_0}$ , the jump of the integral is given by the mean of  $H$  along this curve multiplied by the jump of the semimartingale  $Z_t$ , see [11].

We consider that the number of jumps in the integrator process  $Z_t$  is finite in bounded intervals a.s.. For example, this hypothesis includes the so called Levy-jump diffusion  $Z_t^i = B_t^i + \sum_{k=0}^{N_t} J_k^i$ , where  $B_t^i$  is a Brownian motion,  $N_t$  is a Poisson process and the random variables  $(J_k)$  are i.i.d. (see e.g. Applebaum [1]).

### 3 An extension of Itô-Ventzel-Kunita formula

In the continuous case, we have the well known Itô-Ventzel-Kunita formula: Let  $\varphi_t$  be a stochastic flow of (local) diffeomorphisms, which is solution of the following Stratonovich SDE

$$dx_t = \sum_{i=1}^m X^i(x_t) \circ dN_t^i,$$

where  $N_i$  are continuous semimartingales, and  $X^i$  are smooth vector fields in  $\mathbf{R}^d$ , for  $1 \leq i \leq m$ . Let  $U_t = (U_t^1, \dots, U_t^d)$  be a continuous semimartingale. Then:

**Theorem 3.1.** (*Itô-Ventzel-Kunita formula for continuous case*)

$$\varphi_t(U_t) = \varphi_0(U_0) + \sum_{i=1}^m \int_0^t X^i(\varphi_s(U_s)) \circ dN_s^i + \sum_{j=1}^d \int_0^t \frac{\partial \varphi_s}{\partial x_j}(U_s) \circ dU_s^j. \quad (5)$$

For a proof, see e.g. Kunita [9, Thm 8.3]). A direct corollary of this result is a Leibniz formula for the composition of stochastic flows of diffeomorphisms in the continuous case, that is,  $d(\varphi \circ F)_t = d\varphi_t \circ F_t + (\varphi_t)_* \circ dF_t$ , where the differential form is considered in the Stratonovich sense.

In the proof of the main result of the next section, one needs an extension of this Leibniz formula for flows generated by Stratonovich SDE driven by semimartingales with jumps. Precisely, let  $\psi_t$  and  $\xi_t$  be flows of diffeomorphisms generated by Marcus equations with respect to the same semimartingale with jumps  $Z_t$ :

$$\diamond d\psi_t = X(\psi_t) \diamond dZ_t \quad \text{and} \quad \diamond d\xi_t = Y(\xi_t) \diamond dZ_t.$$

where  $X, Y$  are smooth vectors fields. We assume that  $Z_t$  has the property of finite jumps, as described in the previous section. We define the following integral which generalizes the classical Marcus integral (3), (i.e. [11, Eq. 2.2]):

$$\begin{aligned} \int_0^t \psi_{s*} Y(\xi_s) \diamond dZ_s &:= \int_0^t \psi_{s*} Y(\xi_{s-}) dZ_s + \frac{1}{2} \int_0^t (X'(Y(\xi_s)) + \psi_{s*}(Y'Y)) d[Z, Z]_s^c \\ &+ \sum_{0 \leq s \leq t} \left\{ -\psi_{s-*}(Y(\xi_{s-})) \Delta Z_s - \phi(X \Delta Z_{t_0}, \psi_{t_0-}(\xi_{t_0-})) \right. \\ &\quad \left. + \phi(X \Delta Z_s, \psi_{s-}(\phi(Y \Delta Z_s, \xi_{s-}))) \right\} \end{aligned} \quad (6)$$

The first term on the right hand side of (6) is the standard Itô integral of the predictable process  $\psi_{s*}(Y(\xi_{s-}))$ . The second term corresponds to the finite variation, such that the continuous part of the expression satisfies the classical Itô-Ventzel-Kunita formula. In the summation, we use the notation  $\phi(X, x_0, u)$  for the solution of ordinary differential equation with respect to the vector field  $X$ , and initial condition  $x_0$  at time  $u$  (if omitted, it means that  $u = 1$ ). Note that if  $\psi_s$  is the identity for all  $s \in [0, t]$ , one recovers equation (3).

With the same notation as before, our extension of Itô-Ventzel-Kunita formula is given by

**Theorem 3.2.** (*Itô-Ventzel-Kunita for Stratonovich SDE with jumps*) Suppose that the stochastic flows  $\psi_t$  and  $\xi_t$  are defined in the interval  $[0, a]$ . Then, for all  $t \in [0, a]$ :

$$\psi_t(\xi_t) = \psi_0(\xi_0) + \int_0^t X(\psi_s(\xi_s)) \diamond dZ_s + \int_0^t \psi_{s*}(Y(\xi_s)) \diamond dZ_s. \quad (7)$$

*Proof.* Consider  $t_0 = \sup \{s \in [0, a] \text{ such that the formula holds in } [0, s]\}$ . We want to prove that  $t_0 = a$ . The proof follows by contradiction considering two cases: when the driven process  $Z_t$  is continuous at  $t_0$ , and when it jumps at  $t_0$ . In the first case, if  $t_0 < a$ , there exists a.s. an  $\varepsilon > 0$  such that the process  $Z_t$  is continuous in  $[t_0, t_0 + \varepsilon]$ .

So, by Itô-Ventzel formula for the continuous case (Thm. 3.1), the relation is valid up to time  $t_0 + \varepsilon$ , hence we have a contradiction.

On the other hand, if the process  $Z_t$  jumps at  $t_0$ , and  $t_0 < a$ , we prove that the relation is also valid for  $t_0 + \varepsilon$ , for a positive  $\varepsilon$ . In fact, concatenating the jump of  $\xi_{t_0}$  and the jump of  $\psi_{t_0}$  we have that:

$$\psi_{t_0}(\xi_{t_0}) = \psi_{t_0^-}(\xi_{t_0^-}) + \phi(X\Delta Z_s, \psi_{t_0^-}(\phi(Y\Delta Z_s, (\xi_{t_0^-}))) - \psi_{t_0^-}(\xi_{t_0^-}). \quad (8)$$

Initially use that

$$\psi_{t_0^-}(\xi_{t_0^-}) = \psi_0(\xi_0) + \int_0^{t_0^-} X(\psi_s(\xi_s)) \diamond dZ_s + \int_0^{t_0^-} \psi_{s*}(Y(\xi_s)) \diamond dZ_s.$$

By definition,

$$\int_0^{t_0} X(\psi_s(\xi_s)) \diamond dZ_s = \int_0^{t_0^-} X(\psi_s(\xi_s)) \diamond dZ_s + \phi(X\Delta Z_{t_0}, \psi_{t_0^-}(\xi_{t_0^-})) - \psi_{t_0^-}(\xi_{t_0^-})$$

and

$$\begin{aligned} \int_0^{t_0} \psi_{s*}(Y(\xi_s)) \diamond dZ_s &= \int_0^{t_0^-} \psi_{s*}(Y(\xi_s)) \diamond dZ_s - \phi(X\Delta Z_{t_0}, \psi_{t_0^-}(\xi_{t_0^-})) \\ &\quad + \phi(X\Delta Z_{t_0}, \psi_{t_0^-}(\phi(Y\Delta Z_{t_0}, (\xi_{t_0^-}))))). \end{aligned}$$

Hence, equation (8) yields

$$\psi_{t_0}(\xi_{t_0}) = \psi_0(\xi_0) + \int_0^{t_0} X(\psi_s(\xi_s)) \diamond dZ_s + \int_0^{t_0} \psi_{s*}(Y(\xi_s)) \diamond dZ_s.$$

So, the formula also holds for  $t = t_0$ . By assumption, there exists a. s. an  $\varepsilon > 0$  such that  $Z_t$  is continuous in  $(t_0, t_0 + \varepsilon)$ . Applying again the classical Itô-Ventzel-Kunita formula, we have a contradiction. Hence  $t_0 = a$ .  $\square$

Next corollary contains the Leibniz formula which we are going to use in order to obtain the SDEs for the components of our decomposition in each corresponding Lie subgroup. Its proof follows straightforward from Theorem 3.2.

**Corollary 3.3.** *Let  $\psi_t$  and  $\xi_t$  be flows generated by Marcus equations with respect to the same semimartingale  $Z_t$ , with the property of finite jumps, then:*

$$\diamond d(\psi \circ \xi)_t = \diamond d\psi_t \circ \xi_t + (\psi_t)_* \circ \diamond d\xi_t. \quad (9)$$

## 4 Decomposition of stochastic flows with jumps

In this section, following the framework of [11], we assume that the manifold  $M$  is embedded in an Euclidean space. Hence, smooth vector fields in  $M$  can be extended to smooth vector fields in a tubular neighbourhood of  $M$ . Let  $\varphi_t$  be a stochastic flow of (local) diffeomorphisms in  $M$  generated by a Stratonovich SDE in the sense of

Marcus equation, where the integrator has the property of finite jumps, as described in the previous section.

Assume that the manifold  $M$  has (locally) a pair of differentiable distributions, that we call horizontal and vertical. We denote the first one by  $\Delta^H : U \subseteq M \rightarrow Gr_k(M)$  and the second by  $\Delta^V : U \subseteq M \rightarrow Gr_{n-k}(M)$ . Also, we assume that the horizontal and vertical distributions are complementary, that is, for each  $x \in U$ ,  $\Delta^H(x) \oplus \Delta^V(x) = T_x M$ .

#### 4.1 Existence of the decomposition

We are interested in decomposing the stochastic flow as  $\varphi_t = \xi_t \circ \psi_t$ , where the components  $\xi_t \in \text{Diff}(\Delta^H, M)$  and  $\psi_t \in \text{Diff}(\Delta^V, M)$ . Initially, consider the following definitions:

**Definition 4.1** (Adjoint distribution). *Let  $\Delta : U \subseteq M \rightarrow Gr_p(M)$  be a differential distribution in  $M$ , and take  $\xi \in \text{Diff}(M)$ . The distribution  $\text{Ad}(\xi)\Delta : U \subseteq M \rightarrow Gr_p(M)$  is given by:*

$$\text{Ad}(\xi)\Delta(x) = \xi_*\Delta(\xi^{-1}(x)).$$

**Definition 4.2.** *Let  $\Delta^H$  and  $\Delta^V$  be a pair of locally complementary distributions in  $M$ . We say that  $\Delta^H$  and  $\Delta^V$  preserve transversality along  $\text{Diff}(\Delta^H, M)$  if, for each  $\xi \in \text{Diff}(\Delta^H, M)$ , the distributions  $\Delta^H$  and  $\text{Ad}(\xi)\Delta^V$  are locally complementary.*

Assuming the condition of transversality in the distributions, we have a result on the decomposition of a continuous stochastic flow  $\varphi_t$  generated by a canonical Stratonovich SDE [4, Thm 2.2]:

**Theorem 4.3.** *Let  $\Delta^H$  and  $\Delta^V$  be two complementary distributions in  $M$  which preserve transversality along  $\text{Diff}(\Delta^H, M)$ . Given a continuous stochastic flow  $\varphi_t$  then, up to a stopping time, there exists a factorization  $\varphi_t = \xi_t \circ \psi_t$ , where  $\xi_t$  is a continuous diffusion in  $\text{Diff}(\Delta^H, M)$  and  $\psi_t$  is a continuous process in  $\text{Diff}(\Delta^V, M)$ .*

Initially, a straightforward extension of this result holds for a stochastic flow  $\varphi_t$  generated by a Stratonovich SDE with respect to a semimartingale  $Z_t$  with jumps (proposition below). In the next section we present a stronger result showing the equations of each component in  $\text{Diff}(\Delta^H, M)$  and  $\text{Diff}(\Delta^V, M)$ .

**Proposition 4.4.** *Suppose that the distributions  $\Delta^V$  and  $\Delta^H$  preserves transversality along  $\text{Diff}(\Delta^H, M)$ . Given a stochastic flow  $\varphi_t$  generated by a Stratonovich SDE with jumps, then, up to a stopping time  $\tau$ , there exists a factorization  $\varphi_t = \xi_t \circ \psi_t$ , where  $\xi_t$  is a diffusion in  $\text{Diff}(\Delta^H, M)$  and  $\psi_t$  is a process in  $\text{Diff}(\Delta^V, M)$ .*

*Moreover, if  $t_0$  is a point of discontinuity of  $Z_t$ ,  $t_0 \leq \tau$  and the decomposition holds for  $\varphi_{t_0}$ , then  $t_0 < \tau$ , a.s..*

*Proof.* By assumption, the first discontinuity of the integrator  $Z_t$  happens at a stopping time  $t_1 > 0$  a.s.. Hence, Theorem 4.3 guarantees the existence of the decomposition at least up to  $t_1$ . For the last statement, take  $\delta > 0$  such that the integrator  $Z_t$  is continuous in  $(t_0, t_0 + \delta)$  and apply again Theorem 4.3.  $\square$

In the last statement of the Proposition above, the hypothesis on the decomposition of the diffeomorphism  $\varphi_{t_0}$  cannot be removed: In fact, it may happen that the decomposition holds up to the time  $t_0^-$ , and the process jumps to a non decomposable diffeomorphism  $\varphi_{t_0}$ . As a basic example to illustrate this fact, consider the canonical horizontal and vertical foliations in  $R^2$ . Take the pure rotation system  $dx_t = Ax_t \diamond dZ_t$ , where  $Z_t$  is a semimartingale such that  $Z_t \notin \{\pi/2 + k\pi, k \in \mathbf{Z}\}$  for  $t < t_0$ . The corresponding flow and its decomposition for  $t < t_0$  is given by

$$\varphi_t = \begin{pmatrix} \cos Z_t & -\sin Z_t \\ \sin Z_t & \cos Z_t \end{pmatrix} = \begin{pmatrix} \sec Z_t & -\tan Z_t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin Z_t & \cos Z_t \end{pmatrix}.$$

If at time  $t_0$  the process jumps to  $Z_{t_0} \in \{\pi/2 + k\pi, k \in \mathbf{Z}\}$ , then  $\varphi_{t_0}$  is not decomposable. On the other hand, if the stochastic flow jumps to a decomposable diffeomorphism  $\varphi_{t_0}$ , then the decomposition holds at least an additional  $\delta > 0$  longer than  $t_0$ . In Morgado and Ruffino [16], we have used this fact to construct a stochastic flow  $\tilde{\varphi}_t$  generated by a Stratonovich SDE with controlled jumps that approximates arbitrarily a stochastic flow  $\varphi_t$  generated by a continuous Stratonovich SDE, with the property that  $\tilde{\varphi}_t$  can be decomposable for all  $t > 0$ .

## 4.2 Stratonovich SDEs with jumps for the components

Using the extension of Itô-Ventzel-Kunita formula for Stratonovich SDE with jumps, one can obtain the correspondent SDEs (in the sense of Marcus equation) for the components  $\xi_t$  and  $\psi_t$  in the decomposition above. Initially, we have that  $\varphi_t$  satisfies the following right invariant equation in the Lie group terminology:

$$d\varphi_t = \sum_{i=1}^m R_{\varphi_t*} X^i \diamond dZ_t^i,$$

where  $R_{\varphi_t*}$  is the derivative of the right translation of  $\varphi_t$  in the identity of  $\text{Diff}(M)$ . In fact, for all  $i \in \{1, \dots, m\}$ , it holds that  $R_{\varphi_t*} X^i = X^i(\varphi_t)$ , since if  $\gamma$  is a curve in  $\text{Diff}(M)$  such that  $\gamma(0) = Id$  and  $\gamma'(0) = X^i$ , we have:

$$R_{\varphi_t*}(X^i) = \frac{d}{ds}(\gamma(s) \circ \varphi_t)|_{s=0} = X^i(\varphi_t).$$

By assumption in Proposition 4.4, we consider that  $\Delta^H$  and  $\Delta^V$  preserves transversality along  $\text{Diff}(\Delta^H, M)$ . So, for each  $\xi \in \text{Diff}(\Delta^H, M)$ , the distributions  $\Delta^H$  and  $Ad(\xi)\Delta^V$  are complementary. For each  $x \in U$ , we have a unique decomposition of  $X^i(x)$  in the corresponding subspaces, given by:

$$X_i(x) = \widetilde{X}_i(x) + v_i(\xi, x),$$

where  $\widetilde{X}_i(x) \in \Delta^H(x)$  and  $v_i(\xi, x) \in Ad(\xi)\Delta^V(x)$ . Now, consider the autonomous Stratonovich SDE in the subgroup  $\text{Diff}(\Delta^H, M)$ , given by:

$$dx_t = \sum_{i=1}^m \widetilde{X}_i(x_t) \diamond dZ_t^i,$$



with initial condition  $x_0 = Id$ . We define  $\xi_t$  as the solution of this equation. So, in the Lie group terminology, the stochastic flow  $\xi_t$  satisfies:

$$d\xi_t = \sum_{i=1}^m R_{\xi_t*} \widetilde{X}_i \diamond dZ_t^i. \quad (10)$$

Note that this equation is not right invariant in the Lie group since  $\widetilde{X}_i$  depends on  $\xi_t$ . By support theorem, it follows that  $\xi_t \in \text{Diff}(\Delta^H, M)$ . Finally, define  $\psi_t = \xi_t^{-1} \circ \varphi_t$ . Using the Corollary 3.3, we can write the following equation for  $d\xi_t^{-1}$ :

$$d\xi_t^{-1} = \sum_{i=1}^m -L_{\xi_t^{-1}*} \widetilde{X}_i \diamond dZ_t^i, \quad (11)$$

where  $L_{\xi_t^{-1}*}$  is the derivative of the left translation of  $\xi_t^{-1}$  at the identity of the Lie group  $\text{Diff}(\Delta^H, M)$ . In fact, just take the time derivative of  $\xi_t \circ \xi_t^{-1} = Id$ . Therefore, using again the Corollary 3.3, one can obtain an expression for  $d\psi_t$ , given by:

$$\begin{aligned} d\psi_t &= \sum_{i=1}^m \left( \xi_t^{-1} X_i \xi_t \psi_t - \xi_t^{-1} \widetilde{X}_i \xi_t \psi_t \right) \diamond dZ_t^i = \\ &= \sum_{i=1}^m Ad(\xi_t^{-1}) (X_i - \widetilde{X}_i) \psi_t \diamond dZ_t^i = \\ &= \sum_{i=1}^m Ad(\xi_t^{-1}) v_i(\xi_t) \psi_t \diamond dZ_t^i. \end{aligned} \quad (12)$$

Note that, by construction, we have that  $v_i(\xi_t, x) \in Ad(\xi_t) \Delta^V(x)$ . Hence,  $Ad(\xi_t^{-1}) v_i(\xi_t, x) \in \Delta^V(x)$  for all  $x \in U$ , and using support theorem again, one conclude that  $\psi_t \in \text{Diff}(\Delta^V, M)$ . So, with the same notation as before, we have proved the following result:

**Theorem 4.5.** *The components  $\xi_t$  and  $\psi_t$  in the decomposition above can be described, respectively, as the solutions of the following Stratonovich SDEs with jumps:*

$$d\xi_t = \sum_{i=1}^m \widetilde{X}_i(\xi_t) \diamond dZ_t^i \quad \text{and} \quad d\psi_t = \sum_{i=1}^m Ad(\xi_t^{-1}) v_i(\xi_t)(\psi_t) \diamond dZ_t^i.$$

with initial conditions  $\xi_0 = \psi_0 = Id$ .

### 4.3 Compatibility of the vector fields and the pair of distributions

In some geometrical structures where we have a sort of compatibility of the vector fields with the pair of complementary distributions, it is possible to guarantee that the decomposition holds for all positive time  $t \geq 0$ .

Given an initial condition  $x_0 \in M$ , consider a local coordinate system  $\alpha_0 : U_0 \subset M \rightarrow \mathbf{R}^p \times \mathbf{R}^{n-p}$ , with  $U_0$  an open set of  $M$  containing  $x_0$  and such that  $\alpha_*(\Delta^V(x_0)) = \{0\} \times \mathbf{R}^{n-p}$ . Analogously, at each time  $t \geq 0$ , consider  $\alpha_t : U_t \subset M \rightarrow \mathbf{R}^p \times \mathbf{R}^{n-p}$ , with  $U_t$  an open set of  $M$  containing  $\varphi_t(x_0)$  such that  $\alpha_{t*}(\Delta^V(\varphi_t(x_0))) = \{0\} \times \mathbf{R}^{n-p}$ . With respect to these coordinate systems one writes the original flow as  $\varphi_t = (\varphi_t^1(x, y), \varphi_t^2(x, y))$ .

**Proposition 4.6.** *Suppose that the decomposition  $\varphi_t = \xi_t \circ \psi_t$  holds up to the stopping time  $\tau$  in a neighbourhood of the initial condition  $x_0$ . Writing in coordinates,*

$$\tau = \sup\{t > 0; \det \frac{\partial \varphi_s^2(x, y)}{\partial y} \neq 0 \text{ for all } 0 \leq s \leq t\}.$$

*Proof.* Straightforward by the inverse function theorem: The local decomposition exists if and only if the  $(n - p) \times (n - p)$  submatrix  $\frac{\partial \varphi^2(x, y)}{\partial y}$  is invertible. That is,  $\varphi = (\xi^1(x, y), Id_2) \circ (Id_1, \varphi^2(x, y))$ , where  $Id_1$  and  $Id_2$  are the identities in  $\mathbf{R}^p$  and in  $\mathbf{R}^{n-p}$ , respectively. □

The proposition above says that the hypothesis for the decomposition can be weakened to conditions only on the original flow  $\varphi_t$  instead of using the transversality preserving property (Definition 4.2).

**Corollary 4.7.** *Suppose that the vertical distribution is integrable (a foliation). If the original flow  $\varphi_t$  sends each vertical leaf into a vertical leaf, then the decomposition  $\varphi_t = \xi_t \circ \Psi_t$  holds for all time  $t \geq 0$ .*

*Proof.* Straightforward from the proposition above since, for all  $t \geq 0$ ,

$$\varphi_{t*} = \begin{pmatrix} \frac{\partial \varphi_t^1(x, y)}{\partial x} & 0 \\ \frac{\partial \varphi_t^1(x, y)}{\partial x} & \frac{\partial \varphi_t^2(x, y)}{\partial y} \end{pmatrix},$$

hence  $\det \frac{\partial \varphi_t^2(x, y)}{\partial y} \neq 0$  for all  $t \geq 0$ . □

**Example 1.** Consider in  $\mathbf{R}^n \setminus \{0\}$  the distribution  $\Delta^H$  given by the spherical foliation and the distribution  $\Delta^V$  given by the radial foliation. Then, for this pair of distribution, any linear system  $\varphi_t$  has the decomposition  $\varphi_t = \xi_t \circ \psi_t$  for all  $t \geq 0$  since each radial line is sent to a radial line.

**Example 2.** Most of relevant dynamics in fibre bundles send fibre into another fibre. The fibres generate the vertical distributions, the horizontal distributions are given by connections. Hence the decomposition for these flows with respect to these pairs of distributions exists for all  $t \geq 0$ . A standard example in this context is the linearised flow  $\varphi_{t*} : T_{x_0}M \rightarrow T_{\varphi_t(x_0)}$  in the linear frame bundle  $\pi : BM \rightarrow M$ . By the Corollary above, the decomposition  $\varphi_{t*} = \xi_t \circ \psi_t$  holds for all  $t \geq 0$  since  $\varphi_{t*}$  is an isomorphism between the fibres  $\pi^{-1}(x_0)$  and  $\pi^{-1}(\varphi_t(x_0))$ .

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